Non-commutative Field Theory on S^4

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Abstract

In the previous paper (hep-th/0402010) we proposed a matrix configuration for a non-commutative S^4 (NC4S) and constructed a non-commutative (star) product for field theories on NC4S. In the present paper we will show that any matrix can be expanded in terms of the matrix configuration representing NC4S just like any matrix can be expanded into symmetrized products of the matrix configuration for non-commutative S^2. Then a scalar field theory on NC4S is constructed. Our matrix configuration describes two S^4's joined at the circle and the Matrix theory action contains a projection matrix inside the trace to restrict the space of matrices to that for one S^4.

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1 Introduction

The construction of the non-commutative S^4 (NC4S) has not been straight-forward. In [1] matrix configuration for NC4S was proposed to describe spherical Longitudinal 5-branes (L5-branes) in the context of the Matrix Theory.[2][3] NC4S was also considered to construct an example of finite 4d field theory.[4][5] There is, however, a problem in describing the fluctuating L5-branes and it was concluded that 'fuzzy S^4' is 6 dimensional, $i.e.$, a fuzzy S^2 fibre bundle over a non-associative S^4.[6] [7]

In the above works it was assumed that the non-commutative S^4 algebra is covariant under $SO(5)$. In the flat, non-compact case, however, the Moyal space is not invariant under space rotations except for two dimensions. The Moyal algebra

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \qquad (\mu, \nu = 1, 2, .., d) \tag{1}$$

is not covariant under $SO(d)$ but only under $SO(2) \times SO(2) \cdots$. Let us suppose that there exists an $SO(5)$ covariant non-commutative algebra on S^4. In the large radius limit this algebra is expected to be replaced by a non-commutative algebra for the tangent space around some point on S^4. This algebra would be covariant under $SO(4)$ which is the symmetry around the axis connecting the point and the origin. This contradicts with the non-existence of the $SO(4)$ covariant non-commutative algebra in the flat, non-compact space. Therefore we must abandon the $SO(5)$ covariant formulation.

In the previous paper[8] the present authors proposed a matrix configuration $\hat{X}_0^A$ ($A = 1, .., 5$) for non-commutative S^4 by means of the tensor products of two $SU(2)$ generators, each in the spin j_1 and j_2 representations, respectively. They satisfy the constraint of the sphere.

$$(\hat{X}_0^A)^2 = \hat{R}^2 \, \mathbf{1} \tag{2}$$

The algebra of these matrices is not a Lie algebra and not covariant under $SO(5)$.

The motivation of the above construction was that the non-commutative algebra of S^2 is identified as $SU(2)$ algebra[13] and the correspondence between the generators T^a of $SU(2)$ and the non-commutative coordinates x^a is established[15]; matrices are mapped onto functions on (commutative) S^2 and the matrix multiplication is mapped onto a non-commutative product (star product) of the functions. When the representation of $SU(2)$ is that with spin

j, any matrix of size $N = 2j + 1$ can be expanded into symmetrized products of the generators T^a up to order $2j$. Correspondingly, the functions on S^2 are spanned by polynomials of x^a up to order $2j$. Then the matrices constructed by tensor products of two matrices will be naturally mapped onto functions on a four dimensional manifold and the non-commutative product for this manifold will be obtained by tensor product of those for two S^2's. Because the matrix configuration satisfies the constraint of S^4, (2), it is expected that non-commutative field theory on S^4 will be obtained.

The functions on S^4 and the non-commutative product constructed in [8] have singularity on a circle $((X^4)^2 + (X^5)^2 = R^2)$ on S^4, which is an ambiguity (indeterminateness) of the values of these functions and the product. This is due to the $S^2 \times S^2$ parametrization of S^4 which is used in the construction of the star product. This singularity is so mild that this is not a serious problem in describing non-commutative field theories on S^4; in non-commutative field theories differentiation is a star-commutator and differentiation does not create more singular terms. In the description of a non-commutative field theory in terms of the Matrix theory there are no singularities, because finite dimensional matrices and traces of them are well-defined.

We will show that any matrix of size $N = (2j_1+1)(2j_2+1)$ can be expressed in terms of the matrix S^4 configuration $\hat{X}_0^A$, if and only if j_1 is half an odd integer. In contrast to the S^2 case this is not a polynomial of $\hat{X}_0^A$'s. Instead, a matrix is expanded into products of the functions of $\hat{X}_0^A$ which are in exact correspondence with the functions on S^4 given by the Matrix $\leftrightarrow$ Function correspondence.

Then we will construct a scalar field theory on NC4S. The Matrix theory action contains a projection matrix inside the trace. This is because our parametrization of S^4 actually describes two S^4's and we must restrict the space of the matrices into the subspace describing only one S^4. The cyclic property of the trace is partially broken, but this is not an obstacle for writing the Matrix theory action for the scalar theory. In the case of the gauge theory there will be a restriction on the gaue symmetry.

Organization of the paper is as follows. In sec.2 we will give some properties of the matrix configuration $\hat{X}_0^A$ for S^4. Especially, we show that any matrix can be expanded in terms of the base matrices constructed from the matrix configuration $\hat{X}_0^A$. In sec.3 we will construct a scalar thery on NC4S. There is

some summary in sec.4.

Finally a comment on the notation: throughout this paper we will use the initial lowercase romans a, b, c,... for values $1, 2, 3$ and the middle romans $i, j, .. = 4, 5$, while the capitals $A, B, ..$ will run from 1 to 5.

2   Matrix configuration

In [8] we proposed the following matrix configuration for NC4S.

$$\hat{X}_0^a = \frac{\alpha}{j_1(j_1 + 1)} \, T_{(j_1)}^3 \otimes T_{(j_2)}^a, \qquad (a = 1, 2, 3)$$

$$\hat{X}_0^4 = \frac{\beta}{2j_1(j_1 + 1)} \, T_{(j_1)}^1 \otimes \mathbf{1}_{2j_2+1}, \qquad \hat{X}_0^5 = \frac{\beta}{2j_1(j_1 + 1)} \, T_{(j_1)}^2 \otimes \mathbf{1}_{2j_2+1}. \tag{3}$$

Here $T_{(j)}^a$ $(a = 1, 2, 3)$ are the generators of the spin-j representation of $SO(3)$ and satisfy

$$[T_{(j)}^a, T_{(j)}^b] = i\epsilon_{abc} \, T_{(j)}^c \tag{4}$$

α and β are normalization constants.[1]

This configuration has the following properties.[2]

(1) The matrices (3) satisfies the matrix analogue of the equation for a four-dimensional ellipsoid.

$$\left(\hat{X}_0^a\right)^2 + 4\frac{\alpha^2}{\beta^2}j_2(j_2 + 1) \, \left(\hat{X}_0^i\right)^2 = \hat{R}^2 \, \mathbf{1} \tag{5}$$

Here $\hat{R}$ is a constant defined by

$$\hat{R} = \alpha \, \sqrt{\frac{j_2(j_2 + 1)}{j_1(j_1 + 1)}}. \tag{6}$$

If we choose

$$\beta = 2\alpha \, \sqrt{j_2(j_2 + 1)}, \tag{7}$$

eq (5) becomes eq (2), the constraint of S^4.

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[1] Actually, we mainly discussed only the case of $j_1 = 1/2$ in [8] and in sec.6 of that paper we made a brief comment on general j_1.

[2] This configuration actually describes two S^4's joined at a circle. To obtain non-commutative field theories on S^4 we must perform a projection onto only one S^4. (sec.3)

(2) This configuration satisfies the following algebra.[3]

$$[\hat{X}_0^a, \hat{X}_0^b] = \frac{2\alpha}{\beta^2} j_1(j_1 + 1) \, \epsilon_{abc} \left(\hat{X}_0^c \, [\hat{X}_0^4, \hat{X}_0^5] + [\hat{X}_0^4, \hat{X}_0^5] \, \hat{X}_0^c \right),$$

$$[\hat{X}_0^a, \hat{X}_0^i] = \frac{j_1(j_1 + 1)}{\alpha} \, \epsilon_{abc}\epsilon_{ij} \left(\hat{X}_0^b \, [\hat{X}_0^c, \hat{X}_0^j] + [\hat{X}_0^c, \hat{X}_0^j] \, \hat{X}_0^b \right),$$

$$(\hat{X}_0^a)^2 \, [\hat{X}_0^4, \hat{X}_0^5] + [\hat{X}_0^4, \hat{X}_0^5] \, (\hat{X}_0^a)^2 = \frac{\beta^2}{2\alpha \, j_1(j_1 + 1)} \, \epsilon_{abc} \, \hat{X}_0^a \, \hat{X}_0^b \, \hat{X}_0^c$$

$$\tag{8}$$

This is not covariant under $SO(5)$ but only under $SO(3) \times SO(2)$.

Properties (1) and (2) can be checked by explicit calculation.

(3) Any $N \times N$ matrix $\hat{M}$ ($N = (2j_1 + 1)(2j_2 + 1)$) can be expressed in terms of $\hat{X}_0^A$, if j_1 is half an odd integer. This is analogous to the cases of non-commutative torus[14] and non-commutative S^2[15]. In the case of non-commutative torus any matrix can be written as a linear combination of $P^n Q^m$, where P, Q are matrices satisfying $PQP^{-1}Q^{-1} = \exp(2\pi i/N)\mathbf{1}$. In the case of non-commutative S^2 any matrix can be expanded into symmetrized products of T^a, the generators of $SO(3)$.

Let us note that any $2j + 1 \times 2j + 1$ matrix can be expanded in terms of the products of $T_{(j)}^a$'s. Therefore it is sufficient to express $T_{(j_1)}^a \otimes \mathbf{1}$ and $\mathbf{1} \otimes T_{(j_2)}^a$ in terms of $\hat{X}_0^A$. First of all from the last two eqs of (3) we obtain

$$T_{(j_1)}^i \otimes \mathbf{1} = \frac{2j_1(j_1 + 1)}{\beta} \, \hat{X}_0^{i+3} \qquad (i = 1, 2). \tag{9}$$

We next compute

$$(\hat{X}_0^a)^2 = \left(\frac{\alpha}{j_1(j_1 + 1)} \right)^2 (T_{(j_1)}^3)^2 \otimes (T_{(j_2)}^a)^2 = \frac{\alpha^2 j_2(j_2 + 1)}{(j_1(j_1 + 1))^2} (T_{(j_1)}^3)^2 \otimes \mathbf{1}. \tag{10}$$

By taking the square root of both sides we have

$$T_{(j_1)}^3 \otimes \mathbf{1} = \frac{j_1(j_1 + 1)}{\alpha\sqrt{j_2(j_2 + 1)}} \, \hat{D}. \tag{11}$$

Here the matrix $\hat{D}$ is defined by

$$\hat{D} \equiv \left((\hat{X}_0^a)^2 \right)^{1/2}. \tag{12}$$

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[3]This algebra is new and defferent from the one we gave in [8].

The sign convention of the square root must be specified. We note that the non-vanishing eigenvalues λ of (10) are all doubled. The square root is then defined such that both signs, $\sqrt{\lambda}$ and $-\sqrt{\lambda}$, are all included in the eigenvalues of (11) and (12).

To obtain $\mathbf{1} \otimes T^a_{(j_2)}$ we must divide $\hat{X}^a_0$ by $T^3_{(j_1)} \otimes \mathbf{1}$ and the latter matrix should not have the eigenvalue 0. Hence j_1 must be half an odd integer. In this case we obtain

$$\mathbf{1} \otimes T^a_{(j_2)} = \sqrt{j_2(j_2+1)} \ \hat{X}^a_0 \ \hat{D}^{-1}. \tag{13}$$

To conclude, any $N \times N$ matrix $\hat{M}$ can be expanded in terms of the products of the matrices,

$$\hat{X}^4_0, \quad \hat{X}^5_0, \quad \hat{D}, \quad \hat{X}^a_0 \ \hat{D}^{-1} \quad (a = 1, 2, 3), \tag{14}$$

as long as j_1 is half an odd integer. Among these there are matrices which are not polynomials of $\hat{X}^A_0$.

3 Scalar field theory on NC4S

3.1 Functions and star product on S^4

In the previous paper [8] we introduced the Matrix $\leftrightarrow$ Function correspondence on the NC4S in terms of the similar correspondence on S^2. In this subsection we will recall the results of [8].

We introduced the following correspondence.[4]

$$T^a_{(j_1)} \leftrightarrow j_1 \ \frac{x^a}{r}, \qquad T^a_{(j_2)} \leftrightarrow j_2 \ \frac{y^a}{\rho}. \tag{15}$$

Here x^a and y^a are coordinates on the two S^2's and $r = \sqrt{(x^a)^2}$, $\rho = \sqrt{(y^a)^2}$. Then the configuration (3) is realized by the following coordinates

$$\begin{aligned}
X^a &= \frac{\alpha j_2}{j_1 + 1} \frac{x^3}{r} \frac{y^a}{\rho}, \quad (a = 1, 2, 3) \\
X^4 &= \frac{\beta}{2(j_1 + 1)} \frac{x^1}{r}, \qquad X^5 = \frac{\beta}{2(j_1 + 1)} \frac{x^2}{r}
\end{aligned} \tag{16}$$

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[4]In [8] we considered only the case $j_1 = 1/2$.

Eqs (15), (16) constitute *the Matrix ↔ Function correspondence.*

One finds that

$$\frac{(j_1 + 1)^2}{(\alpha j_2)^2} \, (X^a)^2 + \frac{4(j_1 + 1)^2}{\beta^2} \, (X^i)^2 = 1. \tag{17}$$

For

$$\alpha = \alpha_* \equiv \frac{j_1 + 1}{j_2} \, R, \qquad \beta = \beta_* \equiv 2(j_1 + 1) \, R, \tag{18}$$

where R denote the radius of the commutative S^4, this gives a constraint equation for S^4. Note that these constants do not satisfy (7). This is interpreted as a quantum effect. The discrepancy disappears only in the limit $j_1, j_2 \to \infty$. In this limit the matrix algebra (8) becomes commutative and the limiting geometry is expected to be that of a commutative S^4. In what follows we will choose these values (18) of α, β.

Matrix multiplication is realized on the functions on S^4 by a non-commutative product $\star$[8], which is induced by the non-commutative products $*_x$, $*_y$ on S^2 [9] [10] [11] [12],

$$f(x) *_x g(x) \;=\; f(x)\, g(x) + \sum_{m=1}^{2j_1} \lambda_1^m \, C_m(\lambda_1) J_{a_1 b_1}(x) \cdots J_{a_m b_m}(x)$$
$$\times \partial_{a_1} \cdots \partial_{a_m} \, f(x) \, \partial_{b_1} \cdots \partial_{b_m} \, g(x) \quad (19)$$

and the similar eq for $f(y) *_y g(y)$. Here $\lambda_1 = 1/2j_1$. $f(x)$ and $g(x)$ are functions of x^a/r. $(r = \sqrt{(x^a)^2})$ $C_m(\lambda)$ and $J_{ab}(x)$ are defined by

$$C_m(\lambda) = \frac{\lambda^m}{m!(1 - \lambda)(1 - 2\lambda) \cdots (1 - (m - 1)\lambda)}, \tag{20}$$

$$J_{ab}(x) = r^2 \delta_{ab} - x^a \, x^b + i \, r \epsilon_{abc} \, x^c. \tag{21}$$

In terms of two $*$'s for x^a and y^a we can define a non-commutative product on S^4.

$$F(X) \star G(X) \equiv F(X) \;\; *_x \otimes *_y \;\; G(X) \tag{22}$$

Here $F(X)$ and $G(X)$ must be regarded as functions of x^a and y^a: the factor $*_x$ of $*_x \otimes *_y$ operates on x^a and $*_y$ acts on y^a, respectively. The final expression must be re-expressed in terms of the S^4 coordinate, X^A. The explicit expression of $\star$ for $j_1 = j_2 = 1/2$ representation was worked out in [8].

The functions $F(X)$ on S^4 are the products of those on the two S^2's. They are polynomials of x^a up to order $2j_1$ and those of y^a up to order $2j_2$,

respectively.

$$F(X) = \sum_{m=0}^{2j_1} \sum_{n=0}^{2j_2} W_{a_1...a_m;b_1...b_n} \ r^{-m} \ \rho^{-n}$$
$$\times \ x^{a_1}(X) \cdots x^{a_m}(X) \ y^{b_1}(X) \cdots y^{b_n}(X) \quad (23)$$

Here $W_{a_1...a_m;b_1...b_n}$ is a constant and symmetric under the interchange of the indices, $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$, separately. $(a_i, b_i = 1, 2, 3)$ By solving (16) x^a and y^a are expressed in terms of X^A as

$$x^i(X) = r \ X^{i+3}/R \ (i = 1, 2), \qquad x^3(X) = r \ D(X)/R,$$
$$y^a(X) = \rho \ X^a/D(X) \quad (a = 1, 2, 3). \tag{24}$$

The function $D(X)$ is defined by

$$D(X) \equiv \pm\sqrt{(X^a)^2}. \tag{25}$$

Actually, these functions (24) correspond to the matrices (14) in accord with the Matrix $\leftrightarrow$ Function correspondence (15). In contrast to the matrix case the coordinates y^a and some of the functions on NC4S are ambiguous on a circle

$$\mathcal{C} = \{(X^A) \mid X^1 = X^2 = X^3 = 0, \ (X^4)^2 + (X^5)^2 = R^2\}. \tag{26}$$

The non-commutative product $\star$ is also ambiguous on $\mathcal{C}$.[8]

Let us introduce polar coordinates (θ_1, φ_1) and (θ_2, φ_2) for the two S^2's; (θ_1, φ_1) for x^a and (θ_2, φ_2) for y^a, respectively. By (16) X^A can be parametrized as follows.

$$X^1 = R \ \cos\theta_1 \ \sin\theta_2 \ \cos\varphi_2, \qquad X^2 = R \ \cos\theta_1 \ \sin\theta_2 \ \sin\varphi_2,$$
$$X^3 = R \ \cos\theta_1 \ \cos\theta_2, \qquad X^4 = R \ \sin\theta_1 \ \cos\varphi_1,$$
$$X^5 = R \ \sin\theta_1 \ \sin\varphi_1 \tag{27}$$

Because $(\theta_1, \varphi_1, \theta_2, \varphi_2)$ and $(\pi - \theta_1, \varphi_1, \pi - \theta_2, \varphi_2 + \pi)$ yield the same X^A the parametrization (27) describes two S^4's: the range of θ_1 must be restricted to $0 \le \theta_1 \le \pi/2$ for one S^4, and the remaining range $\pi/2 \le \theta_1 \le \pi$ will correspond to the other S^4. Therefore $S^2 \times S^2$ is devided into $S_+^2 \times S^2$ and $S_-^2 \times S^2$. The manifold described by (27) is two S^4's joined at the circle $\mathcal{C}$. We will call this manifold $S^4 \natural S^4$. $D(X) = R \ \cos\theta_1$ is positive for the first S^4 and negative for the second. To obtain a single S^4 we must restrict the manifold to the portion

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with $D(X) \geq 0$. In what follows we will assume $D(X) \geq 0$. In the case of the matrix configuration the corresponding matrix $\hat{D}$ (12) was defined such that eigenvalues with both signs are included. Therefore we must introduce an appropriate projection matrix when we compute a trace in evaluating the action integral. In the parametrization (27) X^1, X^2 and X^3 vanish at $\theta_1 = \pi/2$ and the variables θ_2, φ_2 become redundant. This is the origin of the singularity of the star product and functions at $\mathcal{C}$ $(D(X) = 0)$.

Since S^4 is embedded in the flat $\mathbb{R}^5$, the standard round metric

$$
\begin{aligned}
ds_4^2 &= (dX^A)^2 \\
&= R^2 \left(d\theta_1^2 + \sin^2\theta_1 \, d\varphi_1^2 \right) + R^2 \, \cos^2\theta_1 \, \left(d\theta_2^2 + \sin^2\theta_2 \, d\varphi_2^2 \right) \quad (28)
\end{aligned}
$$

yields the correct line element. We will equip our S^4 with this metric. The metric (28) in the polar coordinate (θ_i, φ_i) is also singular at $\theta_1 = \pi/2$; the coefficients of $d\theta_2^2$ and $d\varphi_2^2$ vanish at $\theta_1 = \pi/2$. But this singularity is an apparent singularity common to all the polar coordinates. Instead the functions (23) are non-singular in the polar coordinate.

3.2 Volume forms

The volume form of the round metric in the X^A coordinate system is given by

$$
d(\text{volume}) = \frac{R}{X^3} \, dX^1 \wedge dX^2 \wedge dX^4 \wedge dX^5. \quad (29)
$$

Here the coordinates $X^{1,2,4,5}$ are treated as independent variables. This does not coincide with

$$
\begin{aligned}
d(\text{inv volume}) &= \frac{1}{RD^2X^3} \, dX^1 \wedge dX^2 \wedge dX^4 \wedge dX^5 \\
&= (dx^1 \wedge dx^2/rx^3) \wedge (dy^1 \wedge dy^2/\rho y^3), \quad (30)
\end{aligned}
$$

the product of the $SO(3)$ invariant measures on the two S^2's. These volume forms are related by

$$
d(\text{inv volume}) = \frac{1}{R^2 D(X)^2} \, d(\text{volume}). \quad (31)
$$

The difference of the volume forms will result in the breaking of a part of the cyclic property of the integration, which corresponds to the cyclic property of the trace of matrices. However, it is possible to write down the action integral for the scalar field theory.

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3.3 Scalar field theory

As a simple example of non-commutative field theories on S^4 let us consider a scalar field theory. We propose the matrix action for a real scalar theory on NC4S by

$$S_{\text{scalar}} = \frac{1}{(N/2)} \, Tr_+ \left\{ \left(-\frac{1}{2} \, [\hat{X}_0^A, \hat{\Phi}]^2 + V(\hat{\Phi}) \right) \frac{3\hat{D}^2}{R^2} \right\}. \qquad (32)$$

($N = (2j_1 + 1)(2j_2 + 1)$) Here $\hat{\Phi}$ is an N by N hermitian matrix representing a scalar field and $V(\hat{\Phi})$ is a potential. $\hat{D}$ is a matrix defined in (12). The insertion of $\hat{D}^2$ is motivated by the relation (31) of the two volume forms. Tr_+ denotes a trace restricted to the subspace Λ_+ of positive eigenvalues of $\hat{D}$. We also denote the subspace of negative eigenvalues as Λ_-. The whole operation is equivalent to inserting into the trace of the Lagrangian a matrix $\hat{D}_+^2 = \hat{D}^2 \, \hat{P}_+$, where $\hat{P}_+$ is a projection matrix onto Λ_+.

One may think that this is not allowed because this will break the unitary symmetry $\hat{\Phi} \to \hat{U} \, \hat{\Phi} \, \hat{U}^{-1}$ of the action. In the case of a scalar theory the background $\hat{X}_0^A$ has already broken this symmetry. To construct gauge theories we need the unitary symmetry. But $\hat{D}_+^2$ commutes with $U(N/2) \otimes U(N/2)$. This symmetry can be turned into the non-commutative $U(1)$ gauge symmetry. When we consider a $U(m)$ gauge theory, we must enlarge the matrices $\hat{X}_0^A$, $\hat{\Phi}$ by tensor products with m by m matrices and the matrix $\hat{D}_+^2$ must be chosen to be an identity in this extra space.

This projection enforces the restriction to one S^4 of $S^4 \natural S^4$ and must be performed once with the operation of the trace. Since the two terms in the action have the form $Tr_+ (A \ A \ \cdots \ A) \ \hat{D}^2 = Tr \ (A \ A \ \cdots \ A \ \hat{D}_+^2)$ [5], the location of $\hat{D}_+^2$ inside the trace is irrelevant due to the cyclic property of the trace.

Let us work out the explicit form of the action for the case $j_1 = j_2 = 1/2$ and see the effect of the projection onto Λ_+. From (18) the values of α and β are $\alpha = 3R$, $\beta = 3R$. Then the matrix configuration (3) is given by

$$\hat{X}_0^a = \frac{R}{2} \begin{pmatrix} \sigma_a & 0 \\ 0 & -\sigma_a \end{pmatrix}, \quad \hat{X}_0^4 = R \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \hat{X}_0^5 = iR \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \qquad (33)$$

Here σ_a are Pauli matrices and $\mathbf{1}$ is a 2 by 2 identity matrix. $\hat{P}_+$ and $\hat{D}^2$ are

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[5] $A = [\hat{X}_0^A, \hat{\Phi}]$ in the first term of the action and $A = \hat{\Phi}$ in the second.

given by

$$\hat{P}_+ = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{D}^2 = \frac{R^2}{4} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{34}$$

As for $\hat{\Phi}$ we put

$$\hat{\Phi} = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}, \tag{35}$$

where A, C are 2 by 2 hermitian matrices and B a 2 by 2 matrix. By simple calculation we obtain the result.

$$
\begin{aligned}
-Tr_+ \, [\hat{X}_0^a, \hat{\Phi}]^2 \, \hat{D}^2 &= \frac{R^4}{16} \, tr\{-[\sigma_a, A]^2 + (\sigma_a B + B\sigma_a)\,(\sigma_a B + B\sigma_a)^\dagger\}, \\
-Tr_+ \, [\hat{X}_0^4, \hat{\Phi}]^2 \, \hat{D}^2 &= \frac{R^4}{4} \, tr\{-(B - B^\dagger)^2 + (A - C)^2\}, \\
-Tr_+ \, [\hat{X}_0^5, \hat{\Phi}]^2 \, \hat{D}^2 &= \frac{R^4}{4} \, tr\{(B + B^\dagger)^2 + (A - C)^2\}.
\end{aligned} \tag{36}
$$

Here tr is the trace of 2 by 2 matrices. Therefore the kinetic part of the action $\frac{3R^2}{64} tr\{-[\sigma_a, A]^2 + (\sigma_a B + B\sigma_a)\,(\sigma_a B + B\sigma_a)^\dagger + 16 BB^\dagger + 8(A - C)^2\}$ yields a necessary damping factor for all the variables in the path integral. A and C may be regarded as matrices associated with S^4's with $D(X) \geq 0$ and $D(X) \leq 0$, respectively, while B a matrix connecting the two S^4's. The above example shows that only the matrix A has the 'kinetic term', *i.e.* the commutator term. B and C are auxiliarly fields.

According to the Matrix $\leftrightarrow$ Function correspondence, this action can be rewritten in terms of the non-commutative product. The corresponding scalar field on S^4 is denoted by $\Phi(X)$. We define the non-commutative derivative ∇^A by the commutator.

$$\nabla^A \Phi(X) \equiv -i[X^A, \Phi(X)]_\star \tag{37}$$

Here the star-commutator $[\cdot, \cdot]_\star$ is defined by $[F, G]_\star = F \star G - G \star F$. In the Matrix $\leftrightarrow$ Function correspondence the restricted trace Tr_+ is replaced by the volume integration only on one S^4 with $D \geq 0$, *i.e.*, $S_+^2 \times S^2$.

$$\frac{1}{(N/2)} \, Tr_+ \rightarrow \frac{1}{\mathrm{Vol}(S_+^2 \times S^2)} \int_{S_+^2 \times S^2} d(\text{inv volume}) \tag{38}$$

We must use the invariant volume form and it is given by (30). The prefactor of the integral is the inverse of the volume of half $S^2 \times S^2$,

$$\mathrm{Vol}(S_+^2 \times S^2) = \frac{1}{2} \times (4\pi)^2 = 8\pi^2. \tag{39}$$

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The action integral is now given by

$$
\begin{aligned}
S_{\text{scalar}} \\
= \int_{S_+^2 \times S^2} \frac{d(\text{inv volume})}{\text{Vol}(S_+^2 \times S^2)} \left\{ \frac{1}{2} \left(\nabla^A \, \Phi(X) \right) \star \left(\nabla^A \, \Phi(X) \right) + V(\Phi)_\star \right\} \star \frac{3D^2}{R^2}.
\end{aligned}
\tag{40}
$$

Here $V(\Phi)_\star$ is obtained from $V(\Phi)$ by replacing the ordinary products by $\star$. By using eq (31) we replace the integration measure by that of S^4 equipped with the round metric.

$$
S_{\text{scalar}} = \int_{S^4} \frac{d(\text{volume})}{D^2 \, \text{Vol}(S^4)} \left\{ \frac{1}{2} \left(\nabla^A \, \Phi(X) \right) \star \left(\nabla^A \, \Phi(X) \right) + V(\Phi)_\star \right\} \star D^2.
\tag{41}
$$

Here $\text{Vol}(S^4)$ is the volume of S^4.

$$
\text{Vol}(S^4) = \frac{8\pi^2}{3} \, R^4
\tag{42}
$$

Let us notice that D^2 in the denominator of the measure and D^2 in the integrand do not cancel.

Now let us consider the commutative limit $(j_1, j_2 \to \infty, R \text{ fixed})$ In this limit we expect that the action (41) will reduce to

$$
S_{\text{comm}} = \int_{S^4} \frac{d(\text{volume})}{\text{Vol}(S^4)} \left\{ \frac{1}{2} \, G^{AB}(X) \, \partial_A \, \Phi(X) \, \partial_B \Phi(X) + V(\Phi) \right\}.
\tag{43}
$$

Here G^{AB} $(A, B = 1, \ldots, 5)$ and its inverse, G_{AB}, are the metric in the tangent space $T_p(S^4)$ and cotangent space $T_p^*(S^4)$ at $p = (X^A)$ defined by $ds_4^2 = G_{AB} \, dX^A \, dX^B$, $G_{AB} \, G^{BC} = \delta_{AC} - X^A X^C / R^2$, $X^A \, G_{AB} = X^A \, G^{AB} = 0$; G^{AB} is given by

$$
\begin{aligned}
G^{ab} &= \delta^{ab} - \frac{1}{R^2} \, X^a \, X^b, \\
G^{ai} &= G^{ia} = -\frac{1}{R^2} \, X^a \, X^i, \\
G^{ij} &= \delta^{ij} - \frac{1}{R^2} \, X^i \, X^j
\end{aligned}
\tag{44}
$$

We have not proved this yet. The proof will depend on the explicit form of the star product and the derivative ∇^A for general values of j_1, j_2. We hope to report on this in the future.

In the commutative limit the scalar field $\Phi(X)$ has the form (23) with the summations extending to infinity. This field is singular on the circle, $D(X) =$

0, and differentiation may produce stronger singularities. Is the action (43) well-defined? We will show that this integral is finite. Let us note that all the terms in G^{ab} and G^{ai} are proportional to X^a, X^b except for the term δ^{ab} in G^{ab}. Therefore in the kinetic term in (43) the derivatives with respect to X^a appear in the combination $X^a \, \partial_a$. Since the singulatity of Φ is at most the ambiguity $X^c/D(X)$ and this satisfies $X^a \, \partial_a \, (X^c/D) = 0$, this type of differentiation does not create stronger singularities. The term in G^{ab} proportional to δ^{ab} will create a singularity $\delta^{ab} \, \partial_a \, (X^c/D) \, \partial_b \, (X^d/D) = 1/D^2 \, (\delta^{cd} - X^c \, X^d/D^2)$ but this singularity is integrable. Therefore the commutative limit of the non-commutative scalar field theory is well-defined.

Finally, we must comment on the cyclic property of the integral (38). Usually, the trace of matrices $\hat{A}$, $\hat{B}$ satisfies the cyclic property $Tr\hat{A}\hat{B} = Tr\hat{B}\hat{A}$ and by the Matrix $\leftrightarrow$ Function correspondence the integral of the functions $A(X)$, $B(X)$ satisfies the similar property $\int dX A(X) \star B(X) = \int dX B(X) \star A(X)$. Our trace Tr_+, however, does not have this property: $Tr_+ \ \hat{A}\hat{B}\hat{D}^2 \equiv Tr\hat{A}\hat{B}\hat{D}^2_+ \neq Tr_+ \ \hat{B}\hat{A}\hat{D}^2$. The corresponding integral does not have this propery, either. By the same reason as for the trace, Tr_+, this is not an obstacle for writing the action integral for the scalar theory. When we consider correlation functions, however, we must keep the ordering, not just the cyclic ordering, in the correspondence.

$$\frac{1}{(N/2)} \, Tr_+ \ \hat{A}_1\hat{A}_2 \cdots \hat{A}_n\hat{D}^2 \leftrightarrow \int_{S^4} \frac{d(\text{volume})}{D^2\text{Vol}(S^4)} \, A_1(X) \star A_2(X) \star \cdots \star A_n(X) \star D^2$$
(45)

When we consider the gauge theories we need the cyclic property to implement the gauge symmetry. On the Matrix theory side we have $U(N/2) \times U(N/2)$ symmetry as mentioned above. If $\hat{H}$ is a matrix of the form

$$\hat{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix},$$

where the first block is for Λ_+ and the second for Λ_-, we have $Tr_+ \ \hat{A}\hat{H}\hat{D}^2 = Tr_+ \ \hat{H}\hat{A}\hat{D}^2$ for any matrix $\hat{A}$. The matrix $\hat{H}$ keeps the subspaces Λ_+, Λ_- invariant separately. On the function side, since $D(X)$ is nothing but x^3 as one can see from (24), the functions which correspond to the matrix $\hat{H}$ are x^3 and $y^{1,2,3}$. Then for an arbitrary function $A(X)$ on S^4 and a function $H(X) \equiv H(x^3(X), y^a(X))$ which is a polynomial of $x^3(X)$ and $y^a(X)$, we can

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check the following formula by using (19) and (22).

$$\int_{S^4} \frac{d(\text{volume})}{D^2 \text{Vol}(S^4)} \, A(X) \star H(X) \star D^2 = \int_{S^4} \frac{d(\text{volume})}{D^2 \text{Vol}(S^4)} \, H(X) \star A(X) \star D^2 \quad (46)$$

Therefore we can construct a gauge theory action which is invariant under restricted gauge transformations generated by gauge functions which are polynomials of $x^3(X)$ and $y^a(X)$. The remaining gauge symmetry will recover in the commutative limit. The problem of the gauge symmetry needs further investigation.

4 Summary

In this paper we have shown that any matrix of size $N = (2j_1 + 1)(2j_2 + 1)$ can be expressed in terms of the matrix S^4 configuration $\hat{X}_0^A$. Therefore in the Matrix theory version of the non-commutative field theories on S^4 the fluctuations of NC4S can be expanded in terms of $\hat{X}_0^A$.

Then we constructed a non-commutative scalar field theory on S^4. The novel point is the projection matrix inside the trace which restricts the space of matrices to the subspace corresponding to one of the two S^4's. In the description of the non-commutative field theory in terms of the star product this restriction corresponds to $D(X) \geq 0$. The restriction on the cyclic property of the trace and the integration is also discussed.

We are now investigating the large radius limit of the non-commutative field theories on S^4. We found that Moyal deformation of $\mathbb{R}^4$ can be obtained in the vicinity of a point on S^4 in a suitable $j_1, j_2 \to \infty$ limit. Some insight into the geometry of NC4S may be obtained by studying this limit. The result will be reported elsewhere.[16] We considered only the scalar field theory in the present paper but the matrix formulation of the gauge theory on the NC4S is also possible.[8] One of the methods for the study of the matrix version of the non-commutative field theories, including scalar theory and gauge theory, is the numerical simulation.[17] We are planning to perform the numerical analysis of non-commutative gauge theories on S^4. Inclusion of fermionic fields on NC4S is also an interesting problem. The construction of the fermion action on S^4 and the extension to the non-commutative description will be explored. Finally, the extension of the present work to higher dimensional non-commutative S^{2n} will be straightforward and interesting.

Acknowledgments

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